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Geometric Phase Shift for Detection of Gravitational Radiation

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A geometrical phase shift is predicted for a light beam propagating in the field of a gravitational wave. Gravitational radiation detection experiments are proposed using this new effect, and the corresponding estimates are given.

1. INTRODUCTION

Berry (1984) showed that for a quantum system whose Hamiltonian $H(\zeta)$ depends on some parameters ζ^a and which evolves in time in such a way that during the evolution the state of the system traces out a closed curve C in the space of these parameters, the wave function can get an additional geometrical phase $\theta(C)$. This geometric phase depends on the motion of the system in the space of parameters.

A rotation of the polarization vector was predicted (Chiao and Wu, 1986) for a linearly polarized laser beam traveling along a single helically wound optical fiber; the results were experimentally confirmed and their connection with Berry's phase (Tomita and Chiao, 1986) shown. Later it was found (Cai *et al.*, 1990) that Berry's phase has in fact a classical origin and arises from the intrinsic topological structure of Maxwell's theory if the Minkowski space-time is considered as a background. If k is the wave vector of an electromagnetic wave and $\mathbf{e}(\mathbf{k})$ its complex polarization vector, then the condition $\mathbf{k}^2 = \omega^2(\mathbf{k}) = \text{const}$ determines a sphere S^2. The angle $\theta(\mathbf{k})$ of

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rotation of $\mathbf{e}(\mathbf{k})$ when it is parallely transported along a null geodesic with the tangent four-vector k is defined by

$$\delta\theta = i\mathbf{e}_L(\mathbf{k})\cdot d\mathbf{e}_R(\mathbf{k})$$

where

$$\mathbf{e}_R = 2^{-1/2}(\mathbf{e}_1 + i\mathbf{e}_2), \qquad \mathbf{e}_L = 2^{-1/2}(\mathbf{e}_1 - i\mathbf{e}_2), \qquad \mathbf{e}_L \times \mathbf{e}_R = i\mathbf{k}/\omega$$

and $\mathbf{e} = e^{-i\theta}\mathbf{e}_R$. The integral angle is $\theta = \int \delta\theta$, where integration is performed over the path of $\mathbf{k}$ on S^2; thus θ has a geometric origin. When the curve is a closed path C on S^2, the angle $\theta(C)$ is given by $\theta(C) = -\Omega(C)$, where $\Omega(C)$ is the solid angle of the loop C with respect to the center of the sphere. The above expression is essentially a flat space-time expression (Cai *et al.*, 1990), although applicable also to general relativity (Bildhauer, 1990).

In a second example of Berry's phase in optical experiments (Pancharatnam, 1956), the state space is the Poincaré sphere which describes all possible polarization states of light. For this case the direction of the light propagation is fixed and a cycle of changes in polarization states corresponds to a closed curve on the Poincaré sphere (see, e.g., Bhandari and Manuel, 1988; Simon *et al.*, 1988). The phase $\theta(C)$ is known as the Pancharatnam phase (Pancharatnam, 1956) and is given by $\theta(C) = -(1/2)\Omega(C)$.

Here we predict a similar geometrical phase shift for light beams propagating in the field of a gravitational plane wave or pulse of gravitational radiation (1) cyclically from and (after reflection) to an observer, thus being closely related to the Pancharatnam phase, and (2) along a circular fiber of radius R_0. We show that for a light beam orthogonal to the direction of propagation of a gravitational wave, this phase grows proportionally to L/λ, where L is distance between the observer and reflecting system and λ is the characteristic wavelength of the gravitational wave packet. In the second case if $\lambda = \pi R_0$, resonance occurs and the phase shift grows proportionally to m, the number of revolutions of light. For a pulse of gravitational radiation, the relative phase shift is proportional to the characteristic amplitude of the pulse in either of these two cases [for preliminary results see Mitskievich and Nesterov (1995)].

In this paper we use the space-time signature $(+1, -1, -1, -1)$; Greek indices run from 0 to 3 and Latin from 1 to 3; it is essential to remember these notations when the integration by parts is performed [see (4) and, in the Appendix, (A9)].

2. GENERAL RESULTS

A concise description of this phenomenon can be done using the Newman–Penrose formalism, which is applicable to the propagation of light in arbitrary media [we are interested here in the cases of (1) a vacuum and (2) an optical fiber, both in a gravitational field]. The real null Newman–Penrose vectors are $l = (k^0, \mathbf{k})$ (tangent to the light world line) and n ($l \cdot n = 1$); the complex ones are m and $\bar{m}$ ($m \cdot \bar{m} = -1$). We shall consider $m = e^{-i\theta}\mathbf{e}_R$ to be the circular (right) polarization vector. When light propagates along optical fibers (not necessarily geodesically), the Newman–Penrose description is quite essential for differentiation along l, $D = \nabla_l$ being a Newman–Penrose operator. The corresponding equations reduce to

$$Dl = (\epsilon + \bar{\epsilon})l - \bar{\kappa}m - \kappa\bar{m}$$

$$Dn = -(\epsilon + \bar{\epsilon})n + \pi m + \bar{\pi}\bar{m}$$

$$Dm = \bar{\pi}l - \kappa\bar{n} + (\epsilon - \bar{\epsilon})m$$

The coefficients in the right-hand side of these equations are in general complex functions (a bar denoting complex conjugation); $\epsilon - \bar{\epsilon}$ determines torsion and κ the curvature of the spacelike trajectory of light (an optical fiber) (Penrose and Rindler, 1986, pp. 169ff). The change of polarization angle generally reads

$$\theta = i\int \mathbf{e}_L \cdot D\mathbf{e}_R \, d\eta + \int (\bar{\epsilon} - \epsilon) \, d\eta$$

In a vacuum, a light beam propagates geodesically, thus $Dl = 0$, and the equation of the polarization vector transport reads $Dm = 0$ (Penrose and Rindler, 1986); these two equations yield also $Dn = 0$. For a planar optical fiber, $Dm = \bar{\pi}l - \kappa\bar{n}$. Hence in both cases $\bar{m} \cdot Dm = 0$, thus

$$\theta = i\int_\Gamma \mathbf{e}_L \cdot D\mathbf{e}_R \, d\eta \tag{1}$$

the integration being performed along the light world line Γ canonically parametrized by η. For the left polarization one has to exchange subscripts L and R in (1), or, equivalently, to change the sign in the right-hand side of this equation.

Note that under parallel transport along the spacelike geodesics orthogonal to the observer's world line γ, the vector of right (left) polarization $\mathbf{e}_R$ ($\mathbf{e}_L$) does not change, but it changes under the transport along Γ (the null line of light whose characteristics are measured in the course of the proposed experiment). This very fact makes the existence of nontrivial phase θ essential.

If we intend to consider an experiment to detect gravitational radiation, it is natural to connect the Newman–Penrose frame with the Fermi coordinates $X^\mu = \delta^\mu_0 s + \delta^\mu_i \xi^i u$ (see, e.g., Manasse and Misner, 1963; Misner *et al.*, 1973). Here s is the proper time along the observer's geodesic world line γ, u being the proper length parametrizing the (spacelike) geodesic orthogonal to it, with a unit tangent vector ξ on γ. In fact, ξ describes the direction in which such a spacelike geodesic goes, and it possesses only spatial components different from zero (the temporal coordinate $X^0 = T$ is directed along γ). In Fermi coordinates, components of the corresponding orthonormal tetrad $e^{\mu}_{(\nu)}$ parallely transported along the spacelike geodesic are represented as expansions

$$e^{\mu}_{(0)} = \delta^{\mu}_{0} - \frac{1}{2}\,\mathring{R}^{\mu}_{ij0}X^iX^j + \cdots$$

$$e^{\mu}_{(p)} = \delta^{\mu}_{p} - \frac{1}{6}\,\mathring{R}^{\mu}_{ijp}X^iX^j + \cdots$$

and the connection coefficient (Christoffel symbols) take the form

$$\Gamma^{\mu}_{\nu 0} = \mathring{R}^{\mu}_{\nu i 0}X^i + \cdots$$

$$\Gamma^{\mu}_{ij} = -\frac{2}{3}\,\mathring{R}^{\mu}_{(ij)k}X^k + \cdots$$

Quantities of the type of $\mathring{Q}$ are taken on γ.

The radius of convergence of the series is determined by the conditions (Manasse and Misner, 1963)

$$u_0 << \min\left\{\frac{1}{|\mathring{R}_{\alpha\gamma\delta\beta}|^{1/2}}, \frac{|\mathring{R}_{\alpha\gamma\delta\beta}|}{|\mathring{R}_{\alpha\gamma\delta\beta,i}|}\right\} \tag{2}$$

The first condition $u_0 << |\mathring{R}_{\alpha\gamma\delta\beta}|^{-1/2}$ determines the size of V where the curvature has not yet caused spatial geodesics to cross each other. The second condition defines the domain where the curvature does not change essentially. For instance, for gravitational waves with wavelength η the Riemann tensor is $\sim A\,\exp(ik_{\mu}x^{\mu})/\lambda^2$, where A is the dimensionless amplitude. So equation (2) yields

$$u_0 << \min\{\lambda/\sqrt{A}, \lambda\}$$

Generally it is assumed that $A \leq 10^{-18}$. This means that the size of V is restricted by $u_0 << \lambda$. So the application of Fermi coordinates to modern experiments may be very restrictive since λ is often supposed to be of the order of 300 km. Thus to enlarge the range of validity by a factor $1/\sqrt{A}$ (which is about 10^9 in our example) it is necessary to take into account all derivatives of the Riemann tensor.

Covariant derivatives of the Fermi basis can be described as (see the Appendix)

$$\nabla_{\partial\lambda} e^{\mu}_{(\nu)} = -\int_0^u R^{\mu}_{\nu\lambda p}\xi^p \, d\tau + \frac{1}{u}\int_0^u d\tau \int_0^{\tau} \delta^i_{\lambda} R^{\mu}_{\nu i p}\xi^p \, d\tau' + O(R^2) \qquad (3)$$

where integration is performed along a spacelike geodesic orthogonal to the Fermi observer world line γ. This integral formula obviously includes all derivatives of the Riemann tensor. We shall apply equation (3) for the calculation of the phase shift (1).

Using (3), we find from (1)

$$\theta = -i \int_{\Gamma} e^{\mu}_{\mathrm{L}} e^{\nu}_{\mathrm{R}} l^{\lambda} \, d\lambda \int_0^u R_{\mu\nu\lambda p}\xi^p \, d\tau$$
$$+ i \int_{\Gamma} \frac{1}{u} e^{\mu}_{\mathrm{L}} e^{\nu}_{\mathrm{R}} \, d\eta \int_0^u d\tau \int_0^{\tau} R_{\mu\nu i p} l^i \xi^p \, d\tau' + O(R^2)$$

Integrating by parts, we obtain

$$\theta = -i \int_{\Gamma} e^{\mu}_{\mathrm{L}} e^{\nu}_{\mathrm{R}} l^0 \, d\eta \int_0^u R_{\mu\nu 0 p}\xi^p \, d\tau$$
$$+ i \int_{\Gamma} \frac{1}{u} e^{\mu}_{\mathrm{L}} e^{\nu}_{\mathrm{R}} l^i \, d\eta \int_0^u R_{\mu\nu i p}\xi^p \tau \, d\tau + O(R^2) \qquad (4)$$

We consider now a plane weak gravitational wave whose metric tensor is usually written in synchronous coordinates,

$$ds^2 = \eta_{\mu\nu} \, dx^{\mu} \, dx^{\nu} + h_{ab} \, dx^a \, dx^b$$

where a and b run from 1 to 2, while

$$h_{ab} = h_{ab}(t - z), \qquad h_{22} = -h_{11} = h_{+}, \qquad h_{12} = h_{\times}$$

Using the definition of the Riemann tensor

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2}(h_{\nu\lambda,\mu,\sigma} + h_{\mu\sigma,\nu\lambda} - h_{\mu\lambda,\nu,\sigma} - h_{\nu\sigma,\mu,\lambda})$$

we find that in the linearized theory nonzero components of Riemann tensor are

$$R_{3ab3} = R_{0ab0} = -R_{3ab0} = \frac{1}{2}\ddot{h}_{ab}, \qquad R_{\mu 22\nu} = -R_{\mu 11\nu}, \qquad R_{\mu 12\nu} = R_{\mu 21\nu} \qquad (5)$$

where a dot indicates derivative with respect to t. Now equation (4) is readily applicable, and two typical cases emerge: (A) parallely (antiparallely) propagating gravitational and light waves and (B) mutually orthogonal waves. Below we study both cases.

2.1. Parallely (Antiparallely) Propagating Gravitational and Light Waves

We assume that the both waves propagate along the z axis. One can write $l = l^0(1, 0, 0, \pm 1)$, where $l^0 = dT/d\eta$; the upper sign corresponds to the positive direction propagation of light and the lower sign to the negative direction. Let us take

$$e^{\mu}_{R} = \frac{1}{\sqrt{2}}(0, 1, i, 0), \qquad e^{\mu}_{L} = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$

Equation (4) reduces to

$$\theta = -\int_0^T d\tau \int_0^{u(\tau)} R_{120b}\xi^b \, d\tau' \pm \int_0^T \frac{d\tau}{u(\tau)} \int_0^{u(\tau)} R_{123b}\xi^b\tau' \, d\tau' + O(R^2) \quad (6)$$

Applying (5), we find $\theta = 0$, which means absence of a phase shift.

2.2. Orthogonally Propagating Gravitational and Light Waves

Let the gravitational wave propagate along the z axis and the light beam propagate in the (x, y) plane at an angle ϕ to x axis. We suppose

$$\xi = (0, \cos\phi, \sin\phi, 0), \qquad l = l^0(1, \pm\cos\phi, \pm\sin\phi, 0), \qquad l^0 = dT/d\eta \quad (7)$$

$$e^{\mu}_{R} = \frac{1}{\sqrt{2}}(0, \mp\sin\phi, \pm\cos\phi, i), \qquad e^{\mu}_{L} = \frac{1}{\sqrt{2}}(0, \mp\sin\phi, \pm\cos\phi, -i) \quad (8)$$

where the upper sign corresponds to propagation of light from the observer and the lower one to propagation to the observer. Then the phase shift (4) is given by

$$\theta = \pm\int_0^T d\tau \int_0^{u(\tau)} [R_{3120}(\cos^2\phi - \sin^2\phi) + 2R_{3220}\sin\phi\cos\phi] \, d\tau' + \mathcal{O}(R^2) \quad (9)$$

where we used (5). It is convenient to rewrite this equation as

$$\theta = \mp\frac{1}{2}\int_0^T d\tau \int_0^{u(\tau)} (\ddot{h}_{\times}\cos 2\phi + \ddot{h}_{+}\sin 2\phi) \, d\tau + \mathcal{O}(R^2) \quad (10)$$

Taking into account that $X^{\mu} = x^{\mu} + \mathcal{O}(h)$ and $h = h(t - z)$ (h being h_+ or $h_{\times}$), one can write (10) as

$$\theta = \mp\frac{1}{2}\int_0^T [\ddot{h}_\times(\tau)\cos 2\phi + \ddot{h}_+(\tau)\sin 2\phi]u(\tau)\,d\tau + \mathcal{O}(R^2) \tag{11}$$

Let us define $\theta = \alpha_+ \sin 2\phi + \alpha_\times \cos 2\phi$, where α_+ and $\alpha_\times$ correspond to the two independent polarization modes of the gravitational wave (see, e.g., Misner *et al.*, 1973). Then (for α_+ or $\alpha_\times$)

$$\alpha = \mp\frac{1}{2}\int_0^T \ddot{h}(\tau)u(\tau)\,d\tau + \mathcal{O}(R^2) \tag{12}$$

where h is h_+ or $h_\times$. Integration by parts with $du = \pm d\tau$ yields

$$\alpha = \frac{1}{2}[h(\tau) \mp \dot{h}(\tau)u(\tau)]\,|_0^T$$

where, as above, the upper sign corresponds to the propagation of (right polarized) light from the observer, and the lower one to the observer. For the left polarized light the overall sign of the right-hand side of the last two equations should be changed.

Let us consider an experiment where a circularly polarized electromagnetic wave is traveling between the observer and a reflecting system in the (x, y) plane, the gravitational wave propagating in the positive direction of the z axis. If (say, right) polarization does not change in the course of reflection, the phase shift is

$$\Delta\alpha_1 = \frac{1}{2}[h(2T) - h(0) - 2T\dot{h}(T)] \tag{13}$$

and if polarization changes to the left one,

$$\Delta\alpha_2 = \frac{1}{2}[2h(T) - h(2T) - h(0)] \tag{14}$$

(the effects for an initially left polarized beam have inverse signs).

For multiple reflections we have

$$\Delta\alpha_1(N) = \frac{1}{2}\left[h((N+1)T) + h(NT) - h(T) - h(0) - 2T\sum_{m=1}^{N}\dot{h}(mT)\right] \tag{15}$$

$$\Delta\alpha_2(N) = \frac{1}{2}[h(NT) - h((N+1)T) + h(T) - h(0)] \tag{16}$$

where N is the number of reflections.

For a plane monochromatic gravitational wave, $h(t - z) = A\cos[\omega(t - z) + \delta]$, where A is a dimensionless amplitude. In this case (15) and (16) are rewritten as

$$\Delta\alpha_1(N) = A(\Lambda - \sin\Lambda)\,\frac{\sin[(N/2)\Lambda]\,\sin[(N+1)\Lambda/2 + \delta]}{\sin(\Lambda/2)} \tag{17}$$

$$\Delta\alpha_2(N) = 2A\,\sin\left(\frac{\Lambda}{2}\right)\sin\left(\frac{N\Lambda}{2}\right)\cos\left(\frac{N+1}{2}\,\Lambda + \delta\right) \tag{18}$$

Here $\Lambda = 2\pi L/\lambda$, with L the distance between the observer and reflecting systems and λ is the gravitational wavelength.

The average relative phase shift between the right and left polarized light is

$$\sqrt{\langle\Delta\theta_1^2\rangle} = A\left|(\Lambda - \sin\Lambda)\,\frac{\sin[(N/2)\Lambda]}{\sin(\Lambda/2)}\right| \tag{19}$$

$$\sqrt{\langle\Delta\theta_2^2\rangle} = 2A\left|\sin\frac{\Lambda}{2}\,\sin\frac{N\Lambda}{2}\right| \tag{20}$$

$A = (A_+^2 + A_\times^2)^{1/2}$ is the dimensionless amplitude of an unpolarized gravitational wave, and averaging is performed with respect to all polarizations and phases δ_+ and $\delta_\times$.

We shall consider here no phase change in the course of reflection, since this is the only way to obtain a sensible integral effect. If $\Lambda >> 1$, then, for instance, for $N = 1$, we obtain

$$\sqrt{\langle\Delta\theta^2\rangle} = A\Lambda \tag{21}$$

We see that effectively the dimensionless amplitude A grows by a factor Λ. When $N >> 1$, we find that the maximum phase shift

$$\sqrt{\langle\Delta\theta^2\rangle} = 2\pi mNA \tag{22}$$

occurs for $\Lambda = 2\pi m$, where m is integer. If $\Lambda << 1$, we obtain

$$\sqrt{\langle\Delta\theta^2\rangle} = 2A\left|\sin\left(\frac{N\Lambda}{2}\right)\right| \tag{23}$$

and the relative shift takes its maximum value,

$$\sqrt{\langle\Delta\theta^2\rangle} = 2A \tag{24}$$

when $N\Lambda = \pi$.

Another possible experiment involves a scheme similar to that of Braginski and Menskii (1971), but with measurement of the geometric phase shifts for right and left polarizations [and not the frequency shift as in Braginsky and Menskii (1971)] of light traveling along a circular fiber of radius R_0. Let us consider the case when a plane gravitational wave is propagating in the positive direction of the z axis, the fiber lies in the (x, y) plane, and the light propagates in a counterclockwise direction. We suppose

$$\xi = (0, \cos\phi, \sin\phi, 0), \qquad l = l^0(1, -\sin\phi, \cos\phi, 0), \qquad l^0 = dT/d\eta \tag{25}$$

$$e_R^\mu = \frac{1}{\sqrt{2}}(0, \cos\phi, \sin\phi, i), \qquad e_L^\mu = \frac{1}{\sqrt{2}}(0, \cos\phi, \sin\phi, -i) \tag{26}$$

Then the phase shift is given by

$$\begin{aligned}\theta &= R^0 \int_0^T R_{3ab0}\xi^a(\tau)\xi^b(\tau)\,d\tau + \mathcal{O}(R^2)\\ &= -\frac{R_0}{2}\int_0^T \ddot{h}_{ab}(\tau)\xi^a(\tau)\xi^b(\tau)\,d\tau + \mathcal{O}(R^2)\end{aligned} \tag{27}$$

where we used equation (5). Defining $\theta = \alpha_+ + \alpha_\times$, we obtain

$$\alpha_+ = \frac{R_0}{2}\int_0^T \ddot{h}_+(\tau)\cos(2\omega_0\tau)\,d\tau + O(R^2)$$

$$\alpha_\times = -\frac{R_0}{2}\int_0^T \ddot{h}_\times(\tau)\sin(2\omega_0\tau)\,d\tau + O(R^2)$$

where $\omega_0 = 2\pi/T_0$, and T_0 is the period of revolution of light. Integration by parts with $h(\tau) = A\cos(\omega\tau + \delta)$, where, as above, A is dimensionless amplitude, yields

$$\begin{aligned}\alpha_+ = A_+\left(\frac{\omega}{2\omega_0}\right)\Bigg[&\frac{\sin((\omega_0 - \omega/2)T)\cos((\omega_0 - \omega/2)T - \delta_+)}{1 - 2(\omega_0/\omega)}\\ &- \frac{\sin((\omega_0 + \omega/2)T)\cos((\omega_0 + \omega/2)T - \delta_+)}{1 + 2(\omega_0/\omega)}\Bigg]\end{aligned} \tag{28}$$

$$\begin{aligned}\alpha_\times = A_\times\left(\frac{\omega}{2\omega_0}\right)\Bigg[&\frac{\sin((\omega_0 + \omega/2)T)\sin((\omega_0 + \omega/2)T + \delta_\times)}{1 + 2(\omega_0/\omega)}\\ &- \frac{\sin((\omega_0 - \omega/2)T)\sin((\omega_0 - \omega/2)T - \delta_\times)}{1 - 2(\omega_0/\omega)}\Bigg]\end{aligned} \tag{29}$$

For left polarized light, the overall sign of the right-hand side of the last two equations should be changed. The average relative phase shift between right- and left-polarized light is

$$\sqrt{\langle \Delta\theta^2 \rangle} = \frac{A\omega}{2\sqrt{2}\,\omega_0}\left[\frac{\sin^2(\omega_0 + \omega/2)T}{[1 + 2(\omega_0/\omega)]^2} + 2\,\frac{\sin(\omega_0 - \omega/2)T\,\sin(\omega_0 + \omega/2)T\,\cos(2\omega T)}{1 - (2\omega_0/\omega)^2} + \frac{\sin^2(\omega_0 - \omega/2)T}{[1 - 2(\omega_0/\omega)]^2}\right]^{1/2} \qquad (30)$$

When the gravitational wavelength η is equal to πR_0, a resonance occurs leading to $\langle \Delta\theta^2 \rangle^{1/2} = 2^{1/2}\pi m A$, where m is the number of revolutions of light. It is clear that such a detector is effective for high-frequency gravitational radiation, the corresponding R_0 being around 100 km for 10^3 Hz. A considerably smaller detector size could be achieved for a toroidol winding of the fiber, and this case is currently under consideration. However, measurements of the phase shift in a fiber (and in any medium other than a good vacuum) could be virtually impossible due to the random fluctuations, so that we consider here these cases for the sake of completeness only.

3. DISCUSSION AND CONCLUSION

We discuss the possibility of detecting gravitational radiation based on the proposed new effect. Let us consider the case $\Lambda >> 1$. From (21) we obtain $\langle \Delta\theta^2 \rangle^{1/2} \approx 2 \times 10^{-5} A L \nu$, where ν is characteristic frequency of the gravitational radiation in Hz and L is the distance in km. It is clear that such a detector is effective for high-frequency gravitational radiation ($\nu \sim 10^4$ Hz). If, for instance, the reflecting system is placed on the surface of the moon, we have $\langle \Delta\theta^2 \rangle^{1/2} \sim 8A\nu$. For the 5 million-km-long Laser Interferometer Space Antenna (LISA), which would fly in heliocentric orbit (Thorne, 1995a,b), our estimation of the phase shift is $\langle \Delta\theta^2 \rangle^{1/2} \sim 10^2 A\nu$.

Let us compare the experiment proposed here with the standard experiments involving resonant antennas directed to the Virgo cluster and tuned to some 3000 Hz. In this case $A \sim 10^{-20}$ [see the corresponding data in Douglas and Braginsky (1979) and Thorne (1995a,b)]. So the geometrical phase detector with the base earth–moon treats this radiation as if it had an effective magnitude of some 10^{-16} and for LISA as if it had an effective magnitude of some 10^{-14}.

For experiments using a laboratory-size apparatus, $\Lambda << 1$, we see that $\langle \Delta\theta^2 \rangle^{1/2} \sim 2A\,|\sin(N\Lambda/2)|$, and the relative shift takes its maximum value when $N\Lambda = \pi$. For LIGO/Virgo ($L \sim 3 \times 10^5$ cm) interferometers we find that the corresponding frequency of the gravitational wave is $\nu \sim 3 \times 10^4/N$. It is known that for $L \approx 10^5$ cm one could expect about 300 reflections (Douglas and Braginsky, 1979), which is sufficient for detection of continuous waves with 100 Hz $< \nu <$ 10^4 Hz. The advanced LIGO interferometers are expected to have their optimal sensitivity at $\nu \sim$ 100 Hz, and rather good sensitivity all the way from $\nu \sim$ 10 Hz to $\nu \sim$ 500 Hz (Thorne, 1995a,b).

If we are interested in detection of a burst of gravitational radiation with $L << \lambda$ (let its maximum be at the moment of time $NT/2$ and characteristic duration $\tau = NT$), then (15) yields $\langle \Delta\theta^2 \rangle^{1/2} \sim A$ for $N >> 1$ or $N = 1$, where A is the characteristic dimensionless amplitude of the pulse. Realistic estimates for L and N (LIGO/Virgo) are $L \sim 3 \times 10^5$ cm, $10 < N < 300$, while the characteristic frequency is 100 Hz $< \nu <$ 10^4 Hz.

Measuring this effect consists of a comparison of interference patterns for both circular polarizations of a light beam. Similar measurements of a phase shift between opposite senses of circularly polarized light were performed using a nonplanar Mach–Zehnder interferometer (Chiao *et al.*, 1988); for a very clear exposition of theoretical and experimental details concerning the observation of phase shifts due to geometrical and topological effects, see Chiao (1990), without dealing with any gravitational effects.

It is worth commenting on the order of smallness of the dimensionless amplitude of the gravitational wave A in connection with the limits of short gravitational waves. This order is most invariantly attributed to the space-time curvature, which is calculated using the metric tensor, see (5): Riem $\sim A/\lambda^2$. Thus when $\lambda \to 0$, A should also tend to zero if the curvature keeps its order of magnitude unchanged, and in the limit of short gravitational wavelengths (geometric optics in the gravitational sense) the apparent divergence of the predicted effect at small λ's [see equations (17), (21)] is merely spurious: $\langle \Delta\theta^2 \rangle^{1/2} \sim$ Riem λL.

The effect we predict in this paper makes it in principle possible to detect gravitational waves using not an interferometer as a whole, but only one of its arms, since there is a fundamental difference in propagation of the left and right polarizations along one and the same null line of the light. One has merely to separate the light of these different polarizations *after* it has returned from its travel, then transform the (circular) polarization of one of the resulting beams to the opposite one, and finally observe the interference fringes after mixing the beams.

We think that this effect reflects an interaction between the photon's spin and the space-time curvature which is closely related to the well-known Papapetrou–Mathisson effect.

APPENDIX

Here we shall obtain the formula [equation (3) in the text]

$$\nabla_{\partial_\lambda} e^{\mu}_{(\nu)} = -\int_0^u R^{\mu}_{\nu\lambda p}\xi^p \, d\tau + \frac{1}{u}\int_0^u d\tau \int_0^{\tau} d\tau' \, \delta^i_\lambda R^{\mu}_{\nu i p}\xi^p + \mathbb{O}(R^2)$$

Let us start with the Taylor expansion for the tetrad in a world tube surrounding the world line γ of an inertial Fermi observer:

$$e^{\mu}_{(\nu)} = \delta^{\mu}_{\nu} + \frac{d\mathring{e}^{\mu}_{(\nu)}}{du} u + \frac{1}{2!}\frac{d^2\mathring{e}^{\mu}_{(\nu)}}{du^2} u^2 + \frac{1}{3!}\frac{d^3\mathring{e}^{\mu}_{(\nu)}}{du^3} u^3 + \cdots \tag{A1}$$

u is the canonical parameter along spacelike geodesics orthogonal to γ. Using in Fermi coordinates the equation of parallel propagation

$$\frac{de^{\mu}_{(\nu)}}{du} + \Gamma^{\mu}_{\sigma i} e^{\sigma}_{(\nu)} \xi^i = 0 \tag{A2}$$

we rewrite equation (A1) as

$$e^{\mu}_{(\nu)} = \delta^{\mu}_{\nu} - \frac{1}{2!}\mathring{\Gamma}^{\mu}_{\nu i,l} X^i X^l - \frac{1}{3!}\mathring{\Gamma}^{\mu}_{\nu i,l,p} X^i X^l X^p + \cdots + \mathbb{O}(R^2) \tag{A3}$$

The expansion of the connection coefficients is given by

$$\Gamma^{\mu}_{\nu\lambda} = \mathring{\Gamma}^{\mu}_{\nu\lambda,i} X^i + \frac{1}{2!}\mathring{\Gamma}^{\mu}_{\nu\lambda,i,l} X^i X^l + \cdots \tag{A4}$$

Applying equations (A3) and (A4), we obtain the following series for the spatial covariant derivatives of the tetrad:

$$\nabla_{\partial_i} e^{\mu}_{(\nu)} = \mathring{\Gamma}^{\mu}_{\nu i,k} X^k - \mathring{\Gamma}^{\mu}_{\nu(i,k)} X^k + \frac{1}{2!}(\mathring{\Gamma}^{\mu}_{\nu i,k,l} - \mathring{\Gamma}^{\mu}_{\nu(i,k,l)}) X^k X^l$$
$$+ \frac{1}{3!}(\mathring{\Gamma}^{\mu}_{\nu i,k,l,m} - \mathring{\Gamma}^{\mu}_{\nu(i,k,l,m)}) X^k X^l X^m + \mathbb{O}(R^2) \tag{A5}$$

Now using the relations

$$\tfrac{1}{2}\mathring{R}^{\mu}_{\nu k i} X^k = (\mathring{\Gamma}^{\mu}_{\nu i,k} - \mathring{\Gamma}^{\mu}_{\nu(i,k)}) X^k$$
$$\tfrac{2}{3}\mathring{R}^{\mu}_{\nu k i,l} X^k X^l = (\mathring{\Gamma}^{\mu}_{\nu i,k,l} - \mathring{\Gamma}^{\mu}_{\nu(i,k,l)}) X^k X^l$$
$$\tfrac{3}{4}\mathring{R}^{\mu}_{\nu k i,l,m} X^k X^l X^m = (\mathring{\Gamma}^{\mu}_{\nu i,k,l,m} - \mathring{\Gamma}^{\mu}_{\nu(i,k,l,m)}) X^k X^l X^m$$

one can write the expansion (A5) as

$$\nabla_{\partial_i} e^{\mu}_{(\nu)} = \frac{1}{2}\mathring{R}^{\mu}_{\nu ki}X^k + \frac{2}{3!}\mathring{R}^{\mu}_{\nu ki,l}X^kX^l + \frac{3}{4!}\mathring{R}^{\mu}_{\nu ki,l}X^kX^lX^m + \cdots$$

$$+ \frac{n}{(n+1)!}\mathring{R}^{\mu}_{\nu l_1 i, l_2, \ldots, l_n}X^{l_1}X^{l_2}\cdots X^{l_n} + \cdots + \mathcal{O}(R^2) \qquad \text{(A6)}$$

It is convenient to present this series in the form

$$\nabla_{\partial_i} e^{\mu}_{(\nu)} = \sum_{n=0}^{\infty} \frac{d^n(\mathring{R}^{\mu}_{\nu ki}\xi^k)}{du^n}\frac{u^{n+1}}{(n+2)(n!)} + \mathcal{O}(R^2) \qquad \text{(A7)}$$

Straightforward calculation yields the following integral representation of equation (A7):

$$\nabla_{\partial_i} e^{\mu}_{(\nu)} = -\frac{1}{u}\int_0^u R^{\mu}_{\nu ik}\xi^k\tau\, d\tau + \mathcal{O}(R^2) \qquad \text{(A8)}$$

Integrating by parts, we find

$$\nabla_{\partial_i} e^{\mu}_{(\nu)} = -\int_0^u R^{\mu}_{\nu ik}\xi^k\, d\tau + \frac{1}{u}\int_0^u d\tau \int_0^{\tau} d\tau'\, R^{\mu}_{\nu ik}\xi^k + \mathcal{O}(R^2) \qquad \text{(A9)}$$

Now let us calculate the temporal covariant derivative $\nabla_{\partial_0} e^{\mu}_{(\nu)}$. From equations (A2)–(A4) we easily obtain

$$\nabla_{\partial_0} e^{\mu}_{(\nu)} = \mathring{\Gamma}^{\mu}_{\nu 0,k}X^k + \frac{1}{2!}(\mathring{\Gamma}^{\mu}_{\nu 0,k,l} - \mathring{\Gamma}^{\mu}_{\nu k,0,l})X^kX^l$$

$$+ \frac{1}{3!}(\mathring{\Gamma}^{\mu}_{\nu 0,k,l,m} - \mathring{\Gamma}^{\mu}_{\nu k,0,l,m})X^kX^lX^m + \cdots + \mathcal{O}(R^2) \qquad \text{(A10)}$$

where we have taken into account that $\mathring{\Gamma}^{\mu}_{\nu 0} = 0$. Obviously,

$$\nabla_{\partial_0} e^{\mu}_{(\nu)} = \mathring{R}^{\mu}_{\nu k0}X^k + \frac{1}{2!}\mathring{R}^{\mu}_{\nu k0,l}X^kX^l + \frac{1}{3!}\mathring{R}^{\mu}_{\nu k0,l}X^kX^lX^m + \cdots$$

$$+ \frac{1}{n!}\mathring{R}^{\mu}_{\nu l_1 0, l_2, \ldots, l_n}X^{l_1}X^{l_2}\cdots X^{l_n} + \cdots + \mathcal{O}(R^2) \qquad \text{(A11)}$$

or

$$\nabla_{\partial_0} e^{\mu}_{(\nu)} = \sum_{n=0}^{\infty} \frac{d^n(\mathring{R}^{\mu}_{\nu k0}\xi^k)}{du^n}\frac{u^{n+1}}{(n+1)!} + \mathcal{O}(R^2) \qquad \text{(A12)}$$

The corresponding integral representation reads

$$\nabla_{\partial_0} e^{\mu}_{(\nu)} = -\int_0^u R^{\mu}_{\nu 0 k}\xi^k \, d\tau + \mathbb{O}(R^2) \tag{A13}$$

Finally one can combine equations (A9), (A13) and write

$$\nabla_{\partial_\lambda} e^{\mu}_{(\nu)} = -\int_0^u R^{\mu}_{\nu\lambda p}\xi^p \, d\tau + \frac{1}{u}\int_0^u d\tau \int_0^\tau d\tau' \, \delta^i_\lambda R^{\mu}_{\nu i p}\xi^p + \mathbb{O}(R^2) \tag{A14}$$

Noting that the transformation from arbitrary coordinate system to the Fermi one takes the form $x^\mu = \Lambda^\mu_\nu X^\nu + O(\Gamma)$ [or $x^\mu = \xi^\mu u + O(\Gamma)$], one can write equation (A14) in an arbitrary coordinate system as

$$\nabla_{\partial_\lambda} e^{\mu}_{(\nu)} = -\int_0^u R^{\sigma}_{\gamma\delta\rho}(\Lambda^{-1})^{\mu}_{\sigma}\Lambda^{\gamma}_{\nu}\Lambda^{\delta}_{\lambda}\xi^{\rho} \, d\tau$$

$$+ \frac{1}{u}\int_0^u d\tau \int_0^\tau d\tau' \, R^{\sigma}_{\gamma i\rho}(\Lambda^{-1})^{\mu}_{\sigma}\Lambda^{\gamma}_{\nu}\Lambda^{i}_{\lambda}\xi^{\rho} + \mathbb{O}(R^2) \tag{A15}$$

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